Generating multiple chaotic attractors by switching control

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Imagine...

- Given a nonlinear system $\dot{x} = F(x, p)$ and two attractors

What happens if we switch $p$ between $p_1$ and $p_2$?

- $A_1$ with $p = p_1$
- $A_2$ with $p = p_2$
Can we obtain...
or ...

A₁

A₂
Observation
(Simulation Results)

\[ \dot{x} = F(x, p) \]

Figure 1: Sketch of the scheme (3), (4), partition interval for $N = 3$ for the case $[7p_2, 3p_1, 4p_3]$; $m_1 = 7$, $m_2 = 3$ and $m_3 = 4$. a) trajectory partition; b) parameter variance vs time.
Results

Chen’s System

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= (p - a)x_1 - x_1x_3 + px_2 \\
\dot{x}_3 &= x_1x_2 - bx_3
\end{align*}
\]

where

- \( \bar{a} = 35, b = 3 \)
- \( p = [p_1, p_2], \text{ attractor } A^* \)
  - where
    - \( p_1 = 23.014 \) and \( p_2 = 26.05 \)
  - integration step: \( h = 0.001 \)
- \( p = 24.532 \left( \frac{23.014 + 26.05}{2} \right) \)
  - attractor \( A_p \)

Figure 7: Synthesized the chaotic Chen attractor \( A^* \) with \([p_1,p_2], p_1 = 23.014, p_2 = 26.05, T = 75 \) and \( h = 0.001 \). a) Time series; b) Phase portrait of \( A^* \) and \( A_p \) (\( p = 24.532 \), superimposed; c) Poincaré section of \( A^* \) and \( A_p \), superimposed.
Results

- $a=35$, $b=3$
- $p = [2p_2, 1p_1]$, attractor $A^*$
  where
  
  $p_1 = 25.75$ and $p_2=26.25$
- integration step: $h = 0.001$
- $p = 26.083 \left(\frac{2 \times 25.75 + 26.25}{3}\right)$
  attractor $A_p$

Figure 10: Synthesized limit cycle $A^*$ from the Chen system, with $[2p_2, 1p_1]$, $p_1 = 25.75$, $p_2 = 26.25$, $T = 75$ and $h = 0.001$. a) Time series; b) Phase portrait of $A^*$ and $A_p$ ($p = 26.083$), superimposed; c) Histogram of $A^*$ and $A_p$, superimposed.
Lorenz System

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= x_1(p - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - cx_3
\end{align*}
\]

where

- \( a = 10, \ c = 8/3 \)
- \( p = [1p_1, 1p_2], \)
  where
  - \( p_1 = 93 \) and \( p_2 = 100 \)
- integration step: \( h = 0.001 \)

Figure 8: Synthesized the chaotic Lorenz attractor \( A^* \), with \([1p_1, 1p_2], p_1 = 93, p_2 = 100, T = 75 \), and \( h = 0.001 \).

a) Time series; b) Phase portrait of \( A^* \) and \( A_p \) (\( p = 95.5 \)), superimposed; c) Poincaré section of \( A^* \) and \( A_p \), superimposed.
Results

Figure 11: Synthesized limit cycle $A^*$ from the Lorenz system, with $(l_1, l_2), p_1 = 60, p_2 = 96.25, T = 75$ and $h = 0.001$. a) Time series; b) Phase portrait of $A^*$ and $A_p$ ($p = 03$), superimposed; c) Histogram of $A^*$ and $A_p$, superimposed.
Results

Rössler System

\[
\begin{align*}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + ax_2 \\
\dot{x}_3 &= b + x_3(x_1 - p)
\end{align*}
\]

where

- \(a = b = 0.1\)
- \(p = [1p_1, 1p_2]\),
  where
  - \(p_1 = 6\) and \(p_2 = 12.5\)
- integration step: \(h = 0.002\)

Figure 9: Synthesized the chaotic Rössler attractor \(A^*\), with \([1p_1, 1p_2]\), \(p_1 = 6, p_2 = 12.5\), \(T = 200\) and \(h = 0.002\). a) Time series; b) Phase portrait of \(A^*\) and \(A_p\) (\(p = 0.25\)), superimposed; c) Poincaré section of \(A^*\) and \(A_p\), superimposed.
Results

Figure 12: Synthesized limit cycle $A^*$ from the Rössler system with $[1 p_1, 2 p_2]$, $p_1 = 12.5$, $p_2 = 6$, $T = 800$ and $h = 0.01$. a) Time series; b) Phase portrait of $A^*$ and $A_p$ ($p = 8.1(6)$), superimposed; c) Histogram of $A^*$ and $A_p$, superimposed.
Result with more Complex Switching

Figure 3: Synthesized limit cycle $A^*$ from the Chen system, with $[7p_2, 3p_1, 4p_3]$, $p_1 = 23.014$, $p_2 = 24$, $p_3 = 32.0195$, $T = 75$ and $h = 0.001$. a) Phase portraits of $A^*$ and $A_p$ ($p = 26.080$), superimposed; b) Histogram of $A^*$ and $A_p$, superimposed.

- $p = [7p_2, 3p_1, 4p_3]$
- integration step: $h = 0.001$
Results

- $p = [8p_2, 7p_1, 2p_3]$
- $h = 0.001$

Figure 4: Synthesized the chaotic Lorenz attractor $A^*$, with $[8p_2, 7p_1, 2p_3]$, $p_1 = 103$, $p_2 = 125.5$, $p_3 = 130$, $T = 75$ and $h = 0.001$. a) Phase portrait of $A^*$ and $A_p$ ($p = 78.4706$), superimposed; b) Histogram of $A^*$ and $A_p$, superimposed; c) Poincaré section of $A^*$ and $A_p$, superimposed.
Results

- $p = [2p_2, 5p_1, 3p_3]$ 
- $h = 0.001$

Figure 5: Synthesized limit cycle $A^*$ from the Rössler system, with $[2p_2, 5p_1, 3p_3]$, $p_1 = 18$, $p_2 = 25$, $p_3 = 31$, $T = 75$ and $h = 0.001$. a) Phase portrait of $A^*$ and $A_{p_2}$ ($p = 23.3$), superimposed; b) A zoom-in window of the phase portrait c) Poincaré section of $A^*$ and $A_{p_2}$, superimposed; d) Histogram of $A^*$ and $A_{p_2}$, superimposed.
Remarks

- Let $\mathcal{A}$ be the set of all global attractors depending on parameter $\phi$ of the system, and let $\mathcal{P} \subset \mathcal{R}$ be the set of the corresponding admissible values of $\phi$.

- Let $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\} \subset \mathcal{P}$ be a finite order subset of $\mathcal{P}$ containing $N$ different values of $\phi$, which determines the set of attractors $\mathcal{A}_N = \{A_1, A_2, \ldots, A_N\} \subset \mathcal{A}$.

- There exists an attractor $A^*$ generated by the system with switching parameter $\phi$ in $\mathcal{P}_N$ depending upon the rule, $[m_1 \phi(1), m_2 \phi(2), \ldots, m_N \phi(N)]$ where $m_i$ are some positive integers and $\phi$ permutes the subset $\{1, 2, \ldots, N\}$.

- $A^*$ is “identical” to an attractor $A_{\phi}$ with

$$p = \frac{\sum_{k=1}^{N} \phi^{(k)} m_k}{\sum_{k=1}^{N} m_k}$$
Theory
A Class of Nonlinear Systems

Nonlinear continuous-time autonomous and dissipative systems:

\[ \dot{x}(t) = f(x(t)) + p(t/\lambda)Bx(t) \quad (1) \]

- \( x \in \mathbb{R}^n \): state vector
- \( f : \mathbb{R}^n \to \mathbb{R}^n \): sufficiently smooth vector field on \( \mathbb{R}^n \) to assure the existence and uniqueness of solutions
- \( t \in I = [0, \infty) \)
- \( B \in \mathbb{R}^{n \times n} \): a constant matrix
- \( p(t) \): a periodic function with period \( T \) and mean value of \( q \)
Define an average model of (1):

\[ \dot{y}(t) = f(y(t)) + qB_y(t) \]  

\[ (2) \]

- \[ y \in \mathbb{R}^n \] : state vector
- Note: \[ \frac{1}{T} \int_{t}^{t+T} p(u) du = q \]
- \[ s(t) \] : the unique solution of (2) for a given set of initial conditions
**Average Model**

- **Linearize (2) on a neighborhood of** $s(t)$, we have

  \[
  \dot{e}(t) = [F(t) + q \mathbf{B}] e(t) \equiv A_q(t) e(t) \quad (3)
  \]

- $e(t) = y(t) - s(t)$ ; $F(t) = D_f(t)$ : Jacobian of $f$ evaluated at $s(t)$

- $\Gamma_s := \{ e_0 \in \mathbb{R}^n : \lim_{t \to \infty} e(t) = 0 \}$ : the domain of attraction of (2)

- **Similarly, linearizing (1):**

  \[
  \dot{e}(t) = [F(t) + p(t/\lambda) \mathbf{B}] e(t) \equiv A_p(t) e(t) \quad (4)
  \]

- $e(t) = x(t) - s(t)$

- $x \in \Gamma_s$
Theorem

Assuming that (3) be uniformly exponentially stable, i.e.

\[ \exists C > 0, \mu > 0 \text{ such that } \epsilon(t) \leq C \| \epsilon_0 \| \exp(-\mu t) \quad (5) \]

then with \( e_0 = \epsilon_0 \), there exists a positive scalar \( \lambda > 0 \) such that

\[ \lim_{t \to \infty} \| e(t) - \epsilon(t) \| = \delta(\lambda^2) \]

where \( \delta(\lambda^2) \) is an order function.
Experiments
Experiments

- **Lorenz System**
  \[
  \begin{align*}
  \dot{x} &= a(y - x) \\
  \dot{y} &= cy - rx - xz \\
  \dot{z} &= xy - bz
  \end{align*}
  \]

- **Opamp:**
  LF347

- **Analog multiplier:**
  AD633
Attractors

R1a = 1.74KΩ

R1b = 7.16KΩ
Results when R1 is switched
Applications: Cryptanalysis

- **Chaos-Based Cryptosystem**

  
  
  \[
  \begin{aligned}
  \dot{x}_1 &= (25\beta(t) + a)(x_2 - x_1) \\
  \dot{x}_2 &= (b - 35\beta(t))x_1 - x_1x_3 + (29\beta(t) - c)x_2 \\
  \dot{x}_3 &= x_1x_2 - \frac{\beta(t) + d}{3}x_3
  \end{aligned}
  \]

  
  Transmitter:

- \(\beta(t)\): time-varying key function
- Information modulating parameter (eg. \(c\)) or output (\(x_3\))

X.-J. Wu, Chaos 16, 043118, 2006
Return Map

- Slow time-varying key function: Adaptive attack can be employed
- Fast time-varying key function: Return map (based on $x_1$) can reveal the value of mean of $\beta(t)$ and then adaptive observer is designed based on this mean value.

Conclusion

- Attractors can be synthesized by the switching of parameters, illustrated both in simulations and experiments.
- It can be proved with an average model.
- Example of applications: cryptanalysis.
Hong Kong
Thank you!