Model Reduction of Linear and Nonlinear Control Systems

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Model Reduction for Control Systems

- Full Order Model

\[ \dot{x} = f(x, u) \]
\[ y = h(x) \]

\( u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad x \in \mathbb{R}^n, \quad n >> 1 \)
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\[u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad x \in \mathbb{R}^n, \quad n >> 1\]

- Reduced Order Model

\[
\dot{z} = a(z, u) \\
y = c(z)
\]

\[u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad z \in \mathbb{R}^k, \quad k << n\]
Possible Goals

- The reduced order model should have essentially the same input output behaviour as the full order model.
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- A compensator that achieves a desired performance for the reduced order model should also do so for the full order model.
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- The full order model is a compensator that achieves a desired performance for another system and we seek a reduced order compensator that does also.

- We shall focus on the first goal.
Model Reduction of Dynamical Systems

- **Full Order Model**

\[ \dot{x} = f(x) \]
\[ x \in \mathbb{R}^n, \quad n \gg 1 \]
Model Reduction of Dynamical Systems

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Model Reduction of Dynamical Systems

• Full Order Model

\[ \dot{x} = f(x) \]

\[ x \in \mathbb{R}^n, \quad n \gg 1 \]

• Reduced Order Model

\[ \dot{z} = a(z) \]

\[ z \in \mathbb{R}^k, \quad k \ll n \]

• The reduced order model should display the "essential" behaviour of the full order model.
The model reduction problem for dynamical systems can be viewed as one for control systems by adding an input and output,

\[
\begin{align*}
\dot{x} &= f(x) + u \\
y &= x \\
u &\in \mathbb{R}^n, \quad y \in \mathbb{R}^n
\end{align*}
\]
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But because the input and output dimensions are now large, it may be difficult to reduce the model.
Model Reduction of Dynamical Systems

- Separation into slow and fast modes.

\[
\begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_n
\end{bmatrix}
= \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
+ \bar{f}(x_1, \ldots, x_n)
\]

- Spectral Gap: 
  \[0 \geq \lambda_1 \geq \ldots \geq \lambda_k \gg \lambda_{k+1} \geq \ldots \geq \lambda_n\]

- Other approaches: Petrov Galerkin, nonlinear Galerkin, singular perturbations, center manifolds, inertial manifolds.
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- **Spectral Gap:**

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- **Galerkin projection onto** \(x_{k+1} = \cdots = x_n = 0\),

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Model Reduction of Control Systems

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The control may not directly excite the slow modes.

The output may not be sensitive to changes in the slow modes.
Model Reduction of Linear Control Systems

- We shall focus on state space methods which generalize to nonlinear control systems.

\[
\begin{align*}
\dot{x} &= Fx + Gu \\
y &= Hx \\
x(0) &= 0
\end{align*}
\]

\(x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad \lambda(F) < 0\)
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- Since the unforced system is Hurwitz, it defines an input-output map

\[
\mathcal{IO}_n: L^2(-\infty, \infty; \mathbb{R}^m) \rightarrow L^2(-\infty, \infty; \mathbb{R}^p) \\
\mathcal{IO}_n: u(-\infty: \infty) \mapsto y(-\infty: \infty) \\
y(t) = \int_{-\infty}^{t} He^{F(t-s)}Gu(s) \, ds
\]
Minimal Realization Theory

• What is the smallest state dimension necessary to realize a given input-output map?

• The state dimension is minimal if the system is controllable, 
  \[ \text{rank} \left[ G \right] = n, \]

• and observable, 
  \[ \text{rank} \left[ \begin{bmatrix} H \newline HF \newline \vdots \newline HF^{n-1} \end{bmatrix} \right] = n. \]

• Any system can be reduced to one that is minimal.
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The Hankel map takes past inputs to future outputs

\[ \mathcal{H}_n : L^2(-\infty, 0; \mathbb{R}^m) \rightarrow L^2(0, \infty; \mathbb{R}^p) \]

\[ \mathcal{H}_n : u(-\infty : 0) \leftrightarrow y(0 : \infty) \]
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\[ \mathcal{H}_n : u(-\infty : 0) \mapsto y(0 : \infty) \]

It factors through the current state \( x(0) \) and so it is of finite rank hence compact.

\[ u(-\infty : 0) \mapsto x(0) = \int_{-\infty}^{0} e^{-Fs}Gu(s) \, ds \]

\[ y(t) = He^{Ft}x(0) \]
B.C. Moores’s Balanced Realization Theory.

\[ \dot{x} = Fx + Gu \]
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- Find a reduced order linear system with approximately the same input-output map.
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- Moore assumed that
  - \( F, G \) is a controllable pair,
  - \( F, H \) is a observable pair,
  - \( F \) is Hurwitz, \( \lambda(F) < 0 \).
- If the system is uncontrollable and/or unobservable, we can make it so by passing to a minimal realization.
- Hurwitz is needed to insure the existence of the input-output map.
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- Moore’s insight was that we should restrict to the directions that are easy to excite and ignore directions where changes don’t affect the output very much.
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- **Controllablity Function**

\[
\pi_c(x^0) = \inf_{u(-\infty:0)} \frac{1}{2} \int_{-\infty}^{0} |u(t)|^2 \, dt
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subject to the system dynamics and
\[x(-\infty) = 0, \quad x(0) = x^0.\]
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- Observability Function

\[
\pi_o(x^0) = \frac{1}{2} \int_{0}^{\infty} |y(t)|^2 \, dt
\]

subject to the system dynamics and \( x(0) = x^0, \ u(t) = 0, \ t \geq 0. \)
\( \pi_c(x) \) is the minimal input energy needed to excite the system from the zero state to \( x \).
\begin{itemize}
  \item $\pi_c(x)$ is the minimal input energy needed to excite the system from the zero state to $x$.
  \item $F, G$ controllable implies $\pi_c(x)$ is bounded.
  \item $F$ Hurwitz implies $\pi_o(x)$ is positive definite.
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  \item $H, F$ observable implies $\pi_o(x)$ is positive definite.
  \item $\pi_c(x)$ and $\pi_o(x)$ are quadratic functions because the system is linear and the energies are quadratic, $\pi_c(x) = \frac{1}{2} x' P^{-1} c x$, $\pi_o(x) = \frac{1}{2} x' P_o x$.
\end{itemize}
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• $F, G$ controllable implies $\pi_c(x)$ is bounded.

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• $\pi_o(x)$ is the output energy released by the system as it decays from $x$ back to the zero state.
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• \( \pi_c(x) \) and \( \pi_o(x) \) are quadratic functions because the system is linear and the energies are quadratic,

\[
\pi_c(x) = \frac{1}{2} x' P_c^{-1} x, \quad \pi_o(x) = \frac{1}{2} x' P_o x
\]
Balanced Realization Theory.

- $P_c, P_o$ are the unique positive definite solutions of

$$0 = FP_c + P_c F' + GG'$$
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- $\pi_c(x)$ large implies that it takes a lot of input energy to excite the system in the direction $x$ and so this direction might be ignored in a reduced order model.
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- $\pi_o(x)$ small implies that changes in the direction $x$ lead to small changes in the output energy and so this direction might be ignored in a reduced order model.
Balanced Realization Theory.

- $P_c, P_o$ transform differently under a linear change of states coordinates $x = Tz$

\[
\begin{align*}
P_c & \mapsto T^{-1}P_cT'^{-1} \\
P_o & \mapsto T'P_oT
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- $P_oP_c$ is a similarity invariant

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- \( P_c, P_o \) transform differently under a linear change of states coordinates \( x = T z \)
  
  \[
  P_c \mapsto T^{-1} P_c T' \quad \text{and} \quad P_o \mapsto T' P_o T
  \]

- \( P_o P_c \) is a similarity invariant
  
  \[
  P_o P_c \mapsto T' P_o P_c T'^{-1}
  \]

- Its eigenvalues are the squares of the singular values of the Hankel map.
Balanced Realization Theory.

- There is a linear change of state coordinates so that the controllability and observability gramians are diagonal and equal,

\[ P_c = P_o = \begin{bmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & \sigma_n \end{bmatrix} \]
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\[ P_c = P_o = \begin{bmatrix} \sigma_1 & 0 \\ \vdots \\ 0 & \sigma_n \end{bmatrix} \]

- The Hankel singular values can be ordered

\[ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \]
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- In these new state coordinates the system is said to be balanced.
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- If the Hankel singular values are distinct then the balanced coordinates are unique up to changes of signs \( x_i \mapsto -x_i \).
Balanced Truncation

- The reduced model is obtained by Galerkin projection onto the states corresponding to large Hankel singular values because they can be reached with relatively small input energy and they produce relatively large output energy.
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- Suppose $\sigma_k \gg \sigma_{k+1}$.
  Let $x_1$ denote the first $k$ components of $x$
  Let $x_2$ denote the last $n - k$ components.
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• Full Order Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
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- Balanced Truncation obtained by Galerkin projection.

\[
\begin{aligned}
\dot{z} &= F_{11} z + G_1 u \\
y &= H_1 z
\end{aligned}
\]
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- The Hankel singular values of the reduced model are \( \sigma_1, \ldots, \sigma_k \).
Balanced Truncation

- The reduced model is obtained by Galerkin (orthogonal) projection in balanced coordinates.
- In the original coordinates, it is a Petrov Galerkin (oblique) projection.
- The Hankel singular values of the reduced model are $\sigma_1, \ldots, \sigma_k$.
- But the reduced model is not an optimal Hankel norm approximation of the full model because the singular vectors are different. Typically

$$\| \mathcal{H}_n - \mathcal{H}_k \| > \sigma_{k+1}$$

Adamjan-Arov-Krein, Glover
Balanced Truncation

- The reduced model is obtained by Galerkin (orthogonal) projection in balanced coordinates.
- In the original coordinates, it is a Petrov Galerkin (oblique) projection.
- The Hankel singular values of the reduced model are $\sigma_1, \ldots, \sigma_k$.
- But the reduced model is not an optimal Hankel norm approximation of the full model because the singular vectors are different. Typically
  \[ \| \mathcal{H}_n - \mathcal{H}_k \| > \sigma_{k+1} \]

Adamjan-Arov-Krein, Glover

- Glover has shown that for balanced truncation
  \[ \| \mathcal{IO}_n - \mathcal{IO}_k \| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]
Balanced Truncation

- Large and small are relative terms and we need one quadratic form to normalize the other.
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- The eigenvalues of the dynamics play an indirect role. It is very hard to excite the system in a direction corresponding to a very stable eigenvalue and so the controllability function tends to be large in such a direction.
- Moreover, a very stable state direction damps out quickly and so the observability function tends to be small in such a direction.

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- Moreover, a very stable state direction damps out quickly and so the observability function tends to be small in such a direction.
- Hence the very stable directions of the dynamics tend to be ignored in the reduction process.
Suppose we have a linear dynamical system in modal coordinates

\[
\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ \ddots & \ddots \\ 0 & \lambda_n \end{bmatrix} x
\]

\[
0 > \lambda_1 \geq \ldots \geq \lambda_n.
\]
Balanced Reduction

Suppose we have a linear dynamical system in modal coordinates

\[ \dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix} x \]

\[ 0 > \lambda_1 \geq \ldots \geq \lambda_n. \]

As before we add a dummy input and output,

\[ \dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix} x + u, \quad y = x \]
Balanced Reduction

Then

\[ P_c = P_o = \begin{bmatrix} \frac{1}{2\lambda_1} & 0 \\ 0 & \frac{1}{2\lambda_n} \end{bmatrix}, \quad \sigma_i = -\frac{1}{2\lambda_i} \]
Balanced Reduction

Then

\[ P_c = P_o = \begin{bmatrix} -\frac{1}{2\lambda_1} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2\lambda_2} & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots \\ 0 & \cdots & 0 & -\frac{1}{2\lambda_n} \end{bmatrix}, \quad \sigma_i = -\frac{1}{2\lambda_i} \]

So balanced truncation is the usual truncation onto the slow modes.

\[-\frac{1}{2\lambda_1} \geq -\frac{1}{2\lambda_2} \geq \cdots \geq -\frac{1}{2\lambda_n} > 0\]
Example

Chain of three masses connected by springs and dashpots attached to a wall at one end. The input is a force applied to the mass next to the wall and the output is the displacement of the mass at the other end. Assume that each mass is $\mu$, each spring constant is $c$ and each dampening constant is $b$. 
Example

The system is linear,

\[ F = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\frac{2c}{m} & \frac{c}{\mu} & 0 & -\frac{2b}{\mu} & \frac{b}{\mu} & 0 & 0 \\
\frac{c}{\mu} & -\frac{2c}{\mu} & \frac{c}{\mu} & \frac{b}{\mu} & -\frac{2b}{\mu} & \frac{b}{\mu} & 0 \\
0 & \frac{c}{\mu} & -\frac{c}{\mu} & 0 & \frac{b}{\mu} & -\frac{b}{\mu} & 0 \\
\end{bmatrix} \quad G = \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \quad H = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]
Example

If $\mu = 1$, $c = 3$, $b = 0.5$ then after balancing

$$P_c = P_o = \begin{bmatrix}
1.6895 \\
1.4901 \\
0.1404 \\
0.1079 \\
0.0077 \\
0.0076
\end{bmatrix}$$
Example

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\[
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\]

$\sigma_2 = 1.4901 \gg 0.1404 = \sigma_3$

This suggests taking a reduced order model of dimension $k = 2$. 
Example

Full (blue) and Reduced (green) Bode Diagrams

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)
Glover has shown that for balanced truncation

\[ ||\mathcal{IO}_n - \mathcal{IO}_k|| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]
Error Estimate

Glover has shown that for balanced truncation

\[ \| \mathcal{IO}_n - \mathcal{IO}_k \| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]

We know that

\[ \sigma_{k+1} \leq \| \mathcal{H}_n - \mathcal{H}_k \| \leq \| \mathcal{IO}_n - \mathcal{IO}_k \| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]
Glover has shown that for balanced truncation

\[ \|IO_n - IO_k\| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]

We know that

\[ \sigma_{k+1} \leq \|H_n - H_k\| \leq \|IO_n - IO_k\| \leq 2 \sum_{j=k+1}^{n} \sigma_j \]

For the spring mass example with \( k = 2 \) this yields

\[ \sigma_3 = 0.1404 \leq \|H_n - H_k\| \leq \|IO_n - IO_k\| \leq 0.5672 \]
Error Estimate

If we restrict the Hankel maps to optimal inputs of the full system then

\[ \| \mathcal{H}_n - \mathcal{H}_k \| \leq 0.1432 \]

Notice how close this is to \( \sigma_3 = 0.1404 \).
Error Estimate

If we restrict the Hankel maps to optimal inputs of the reduced system then

\[ \|H_n - H_k\| \leq 0.0867 \]

Notice how much smaller this is than \( \sigma_3 = 0.1404 \).
Balanced Truncation

- Balanced truncation does not minimize the difference between the input-output maps of the full and reduced models.
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- So what problem does it solve? And how can it be generalized to nonlinear systems?
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• Nonlinear balanced truncation of Scherpen.

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• Differential eigenstructure of nonlinear Hankel maps, Fujimoto and Scherpen.

• Here is a new way of viewing and generalizing linear balanced truncation.
Balanced Truncation

**Full Order Model**

\[
\begin{align*}
\dot{x} &= Fx + Gu \\
y &= Hx
\end{align*}
\]
Balanced Truncation

Full Order Model

\[ \dot{x} = Fx + Gu \]
\[ y = Hx \]

We restrict to those reduced order models that can be obtained by Petrov-Galerkin projection. For this we need an injection \( \Psi \) and a surjection \( \Phi \)

\[ \Psi : \mathbb{R}^k \rightarrow \mathbb{R}^n \]
\[ \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k \]

\[ \Psi : z \mapsto x = \Psi z \]
\[ \Phi : x \mapsto z = \Phi x \]

\[ \Phi \Psi z = z, \quad (\Psi \Phi)^2 x = \Psi \Phi x \]
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Reduced Order Model

\[ \dot{z} = \Phi F \Psi z + \Phi Gu \]
\[ y = H \Psi z \]
Balanced Truncation

**How should we choose the injection** \( \Psi \)?

We choose \( \Psi : \mathbb{R}^k \rightarrow \mathbb{R}^n \) so that its \( k \)-dimensional range maximizes output energy \( \pi_o(x) \) for given input energy \( \pi_c(x) \).

To do this it is convenient to put the system in input normal form, that is, make a linear change of state coordinates so that

\[
\pi_c(x) = \frac{1}{2} \sum_i x_i^2, \quad \pi_o(x) = \frac{1}{2} \sum_i \tau_i x_i^2,
\]

This is just a diagonal change from balanced coordinates and \( \tau_i = \sigma_i^2 \) are the squared Hankel singular values.

If \( \tau_k \gg \tau_{k+1} \) then we should take the range of \( \Psi(z) \) to be \( x_{k+1} = \cdots = x_n = 0 \), e.g.,

\[
\Psi(z_1, \ldots, z_k) = x = (z_1, \ldots, z_k, 0, \ldots, 0)
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We choose $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ so that its $k$ dimensional range maximizes output energy $\pi_o(x)$ for given input energy $\pi_c(x)$. 
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How should we choose the surjection $\Phi$?

We choose $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ to minimize the $L^2$ norm of the difference between the outputs from $x$ and $\Psi(\Phi(x))$. 

Define the co-observability function

$$\pi_{oo}(x, \bar{x}) = \frac{1}{2} \int_{0}^{\infty} \left| y(t) - \bar{y}(t) \right|^2 dt$$

where $y(t), \bar{y}(t)$ are the outputs from $x(0) = x, \bar{x}(0) = \bar{x}$.

Because the system is linear $\pi_{oo}(x, \bar{x}) = \pi_o(x - \bar{x})$.

If the system is in input normal form then the optimal $\Phi$ is

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Nonlinear Balancing

Scherpen generalized Moore to nonlinear systems

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{align*}
\]
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She defined the controllability function,

\[ \pi_c(x^0) = \inf \frac{1}{2} \int_{-\infty}^{0} |u(t)|^2 \, dt \]
subject to the system dynamics and \( x(-\infty) = 0, \ x(0) = x^0. \)
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subject to the system dynamics and \( x(-\infty) = 0, \ x(0) = x^0 \).

And the observability function,

\[ \pi_o(x^0) = \frac{1}{2} \int_{0}^{\infty} |y(t)|^2 \, dt \]

subject to the system dynamics and \( x(0) = x^0, \ u(t) = 0 \).
The controllability function $\pi_c(x)$ and the optimal control $u = \kappa(x)$ satisfy the HJB PDE

\[
0 = \frac{\partial \pi_c}{\partial x}(x) f(x, \kappa(x)) - \frac{1}{2} |\kappa(x)|^2
\]

\[
0 = \frac{\partial \pi_c}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) - \kappa'(x)
\]
Nonlinear Balancing

The controllability function $\pi_c(x)$ and the optimal control $u = \kappa(x)$ satisfy the HJB PDE

$$0 = \frac{\partial \pi_c(x)}{\partial x} f(x, \kappa(x)) - \frac{1}{2} |\kappa(x)|^2$$

$$0 = \frac{\partial \pi_c(x)}{\partial x} \frac{\partial f}{\partial u}(x, \kappa(x)) - \kappa'(x)$$

The observability function $\pi_o(x)$ satisfies the Lyapunov PDE

$$0 = \frac{\partial \pi_o(x)}{\partial x} f(x, 0) + \frac{1}{2} h'(x) h(x).$$
Suppose

- The system is smooth with Taylor expansion

\[
\begin{align*}
\dot{x} & = f(x, u) = Fx + Gu + f^{[2]}(x, u) + \ldots \\
y & = h(x) = Hx + h^{[2]}(x) + \ldots
\end{align*}
\]

where \([d]\) denotes a vector field that is a homogeneous polynomial of degree \(d\).
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where \([d]\) denotes a vector field that is a homogeneous polynomial of degree \(d\).

- The linear part of the system is Hurwitz, controllable and observable
Nonlinear Balancing

Then

- there exist smooth, positive definite local solutions to the above PDEs around $x = 0$
Nonlinear Balancing

Then

• there exist smooth, positive definite local solutions to the above PDEs around $x = 0$

$$
\pi_c(x) = \frac{1}{2} x' P_c^{-1} x + \pi_c^3(x) + \ldots
$$

$$
\pi_o(x) = \frac{1}{2} x' P_o x + \pi_o^3(x) + \ldots
$$
Nonlinear Balancing

Then

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- 
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  \]

- \( P_c, P_o \) are the controllability and observability gramians of the linear part of the system.
Scherpen showed that there is a local change of coordinates that brings the system into the form

$$\pi_c(x) = \frac{1}{2} x' x, \quad \pi_o(x) = \frac{1}{2} x' \begin{bmatrix} \tau_1(x) & 0 \\ \vdots & \ddots \\ 0 & \tau_n(x) \end{bmatrix} x$$
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The $\tau_i(x)$ are the squared singular value functions,

$$\tau_i(0) = \tau_i = \sigma_i^2$$

where $\sigma_i$ are the Hankel singular values of the linear part of the system.
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Unfortunately neither these coordinates nor the squared singular value functions are unique.
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Unfortunately neither these coordinates nor the squared singular value functions are unique.

She obtained a reduced order model by Galerkin projection onto the states with large \( \tau_i(x) \).
The functions $\pi_c(x)$ and $\pi_o(x)$ have power series expansions

\[
\pi_c(x) = \frac{1}{2} x' P_c^{-1} x + \pi_c^{[3]}(x) + \pi_c^{[4]}(x) + \ldots
\]

\[
\pi_o(x) = \frac{1}{2} x' P_o x + \pi_o^{[3]}(x) + \pi_o^{[4]}(x) + \ldots
\]
Input Normal Form of Degree One

The functions $\pi_c(x)$ and $\pi_o(x)$ have power series expansions

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$$

$$
\pi_o(x) = \frac{1}{2} x' P_o x + \pi_o^3(x) + \pi_o^4(x) + \ldots
$$

Following Moore and Scherpen we can make a linear change of coordinates so that

$$
P_c = \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & 1 \end{bmatrix}, \quad P_o = \begin{bmatrix} \tau_1 & 0 \\ \vdots & \ddots \\ 0 & \tau_n \end{bmatrix}
$$

where $\tau_i = \sigma_i^2$ and $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_n > 0$. After this linear change of coordinates the system is said to be in input normal form of degree one.
For simplicity of exposition we shall assume that the $\tau_i$ are distinct, $\tau_1 > \tau_2 > \ldots > \tau_n > 0$. 
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Now

\[
\pi^{[3]}_{c}(x) = \sum_{i \leq j \leq k} \gamma_{c}^{ijk} x_i x_j x_k
\]

\[
\pi^{[3]}_{o}(x) = \sum_{i \leq j \leq k} \gamma_{o}^{ijk} x_i x_j x_k
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For simplicity of exposition we shall assume that the $\tau_i$ are distinct, $\tau_1 > \tau_2 > \ldots > \tau_n > 0$.

Now

$$\pi^c_3(x) = \sum_{i\leq j \leq k} \gamma^i_j^k x_i x_j x_k$$

$$\pi^o_3(x) = \sum_{i\leq j \leq k} \gamma^i_j^k x_i x_j x_k$$

Choose three indices $1 \leq r \leq s \leq t \leq n$, at least two indices are different, $r < t$. Consider the change of coordinates

$$x_r = \xi_r + \beta_r \xi_s \xi_t$$
$$x_t = \xi_t + \beta_t \xi_r \xi_s$$
$$x_l = \xi_l \quad \text{otherwise}$$
Input Normal Form of Degree Two

Then the quadratic parts of $\pi_c$, $\pi_o$ are left unchanged but the cubic parts each pick up an extra term,

$$
\pi_c^{[3]}(\xi) = \sum_{i \leq j \leq k} \gamma_c^{ijk} \xi_i \xi_j \xi_k + (\beta_r + \beta_t)\xi_r \xi_s \xi_t
$$

$$
\pi_o^{[3]}(\xi) = \sum_{i \leq j \leq k} \gamma_o^{ijk} \xi_i \xi_j \xi_k + (\tau_r \beta_r + \tau_t \beta_t)\xi_r \xi_s \xi_t
$$
Input Normal Form of Degree Two

Then the quadratic parts of $\pi_c$, $\pi_o$ are left unchanged but the cubic parts each pick up an extra term,

$$
\pi^{[3]}_c (\xi) = \sum_{i \leq j \leq k} \gamma^i_j_k^c \xi_i \xi_j \xi_k + (\beta_r + \beta_t) \xi_r \xi_s \xi_t
$$

$$
\pi^{[3]}_o (\xi) = \sum_{i \leq j \leq k} \gamma^i_j_k^o \xi_i \xi_j \xi_k + (\tau_r \beta_r + \tau_t \beta_t) \xi_r \xi_s \xi_t
$$

Since $\tau_r > \tau_t$ we can solve the linear system

$$
\begin{bmatrix}
1 & 1 \\
\tau_r & \tau_t
\end{bmatrix}
\begin{bmatrix}
\beta_r \\
\beta_t
\end{bmatrix} =
-\begin{bmatrix}
\gamma^r_s_t^c \\
\gamma^r_s_t^o
\end{bmatrix}
$$
Input Normal Form of Degree Two

Then the quadratic parts of \( \pi_c, \pi_o \) are left unchanged but the cubic parts each pick up an extra term,

\[
\begin{align*}
\pi^{[3]}_c(\xi) &= \sum_{i \leq j \leq k} \gamma_{i,j,k}^{c} \xi_i \xi_j \xi_k + (\beta_r + \beta_t) \xi_r \xi_s \xi_t \\
\pi^{[3]}_o(\xi) &= \sum_{i \leq j \leq k} \gamma_{i,j,k}^{o} \xi_i \xi_j \xi_k + (\tau_r \beta_r + \tau_t \beta_t) \xi_r \xi_s \xi_t
\end{align*}
\]

Since \( \tau_r > \tau_t \) we can solve the linear system

\[
\begin{bmatrix}
1 & 1 \\
\tau_r & \tau_t
\end{bmatrix}
\begin{bmatrix}
\beta_r \\
\beta_t
\end{bmatrix}
= -
\begin{bmatrix}
\gamma_{r,s,t}^{c} \\
\gamma_{r,s,t}^{o}
\end{bmatrix}
\]

This change of coordinates cancels the monomials \( \xi_r \xi_s \xi_t \) from \( \pi^{[3]}_c(\xi), \pi^{[3]}_o(\xi) \).
Input Normal Form of Degree Two

But if \( r = s = t \) then we can cancel the monomial \( x_r^3 \) from only one of \( \pi_c^{[3]}(\xi) \), \( \pi_o^{[3]}(\xi) \) by a change of coordinates of the form

\[
\begin{align*}
  x_r &= \xi_r + \beta_r \xi_r^2 \\
  x_l &= \xi_l \quad \text{otherwise}
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\]
But if $r = s = t$ then we can cancel the monomial $x_r^3$ from only one of $\pi_c^3(\xi)$, $\pi_o^3(\xi)$ by a change of coordinates of the form

$$
\begin{align*}
x_r &= \xi_r + \beta_r \xi^2_r \\
x_l &= \xi_l \quad \text{otherwise}
\end{align*}
$$

We do so in $\pi_c^3(\xi)$ to obtain input normal form of degree two,

$$
\begin{align*}
\pi_c^3(\xi) &= 0 \\
\pi_o^3(\xi) &= \sum_i \gamma_{i}^{iii} \xi_i \xi_i \xi_i
\end{align*}
$$
Input Normal Form of Degree Two

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\begin{align*}
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\end{align*}
\]

If the three indices are distinct \( r < s < t \) then there are many ways to cancel \( x_r x_s x_t \) from \( \pi_c, \pi_o \).
Input Normal Form of Degree $d$

We can do similarly for higher degrees and in this way bring the system into input normal form of degree $d$,

$$\pi_c(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + O(x)^{d+2}$$

$$\pi_o(x) = \frac{1}{2} \sum_{i=1}^{n} \tau_i^{[0:d-1]}(x_i) x_i^2 + O(x)^{d+2}$$

where the squared singular value polynomials $\tau_i^{[0:d-1]}(x_i)$ are of degrees 0 through $d - 1$

$$\tau_i^{[0:d-1]}(x_i) = \tau_i + \tau_{i,1} x_i + \ldots + \tau_{i,d-1} x_i^{d-1}$$
Input Normal Form of Degree $d$

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$$\tau_{i}^{[0:d-1]}(x_i) = \tau_i + \tau_{i,1} x_i + \cdots + \tau_{i,d-1} x_i^{d-1}$$

We have "simultaneously diagonalized" $\pi_c(x), \pi_o(x)$ through terms of degree $\leq d + 1$. There are no cross terms, $x_i x_j \ldots$
The squared singular value polynomial $\tau_i^{[0:d-1]}(x_i)$ measures the relative importance of the state coordinate $x_i$. 
Input Normal Form of Degree $d$

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There may be several ways to bring the system to input normal form of degree $d$ but if $d \leq 6$ then the $\tau_i^{[0:d-1]}(x_i)$ are unique.
The squared singular value polynomial $\tau_i^{[0:d-1]}(x_i)$ measures the relative importance of the state coordinate $x_i$.

There may be several ways to bring the system to input normal form of degree $d$ but if $d \leq 6$ then the $\tau_i^{[0:d-1]}(x_i)$ are unique.

If the system is odd

$$f(-x, -u) = -f(x, u)$$

$$h(-x) = -h(x)$$

then $\pi_c(x)$, $\pi_o(x)$ are even and the $\tau_i^{[0:d-1]}(x_i)$ are unique for $d \leq 12$. 

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Nonlinear Model Reduction

As with linear balanced truncation we restrict to reduced order models that can be found by nonlinear Galerkin projection.
Nonlinear Model Reduction

As with linear balanced truncation we restrict to reduced order models that can be found by nonlinear Galerkin projection.

For this we need an embedding $\psi$ and a submersion $\phi$

\[
\psi : \mathbb{R}^k \rightarrow \mathbb{R}^n \\
\psi : z \mapsto x = \psi(z) \\
\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k \\
\phi : x \mapsto z = \phi(x)
\]

$\phi(\psi(z)) = z, \quad (\psi \circ \phi)^2(x) = (\psi \circ \phi)(x)$
Nonlinear Model Reduction

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\phi(\psi(z)) &= z, \\
(\psi \circ \phi)^2(x) &= (\psi \circ \phi)(x)
\end{align*}
\]

Reduced Order Model

\[
\begin{align*}
\dot{z} &= \frac{\partial \phi}{\partial x}(\psi(z)) f(\psi(z), u) \\
y &= h(\psi(z))
\end{align*}
\]
Nonlinear Model Reduction

We would like to choose the embedding $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ so the $k$ dimensional submanifold that is its range "maximizes" the output energy $\pi_o(x)$ for given input energy $\pi_c(x)$. 
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If \( k > 1 \) then this prescription is not mathematically well-defined so we shall settle for a submanifold that approximates it.
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Assume the system is in input normal form of degree $d$, that the range of input energy of interest is

$$\pi_c(x) \approx \frac{1}{2} |x|^2 \leq \frac{1}{2} c^2$$

and that

$$\tau_i^{[0:d-1]}(x_i) \gg \tau_j^{[0:d-1]}(x_j)$$

for $1 \leq i \leq k < j \leq n$ and $|x_i|, |x_j| \leq c$. 
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for $1 \leq i \leq k < j \leq n$ and $|x_i|, |x_j| \leq c$.

Then a "reasonable" choice of $\psi(z) = x$ is

$$\psi(z_1, \ldots, z_k) = x = (z_1, \ldots, z_k, 0, \ldots, , 0)$$
Nonlinear Model Reduction

How should we choose the submersion $\phi$?
Nonlinear Model Reduction

How should we choose the submersion $\phi$?

We choose $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ to minimize the $L^2$ norm of the difference of the outputs from $x$ and $\psi(\phi(x))$. 

Define the co-observability function as before 

$$
\pi_{oo}(x, \bar{x}) = \frac{1}{2} \int_0^\infty |y(t) - \bar{y}(t)|^2 \, dt
$$

where $y(t), \bar{y}(t)$ are the outputs from $x(0) = x, \bar{x}(0) = \bar{x}$ when $u(t) = 0$.

Then $\pi_{oo}$ satisfies the Lyapunov PDE

$$
0 = \frac{\partial \pi_{oo}}{\partial (x, \bar{x})}(x, \bar{x}) \begin{bmatrix} f(x, 0) & f(\bar{x}, 0) \end{bmatrix} + \frac{1}{2} |h(x) - h(\bar{x})|^2
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As before $\pi_{oo}$ has a power series expansion

$$\pi_{oo}(x, \bar{x}) = \frac{1}{2} \sum_i \tau_i (x_i - \bar{x}_i)^2 + \pi_{oo}^{[3]}(x, \bar{x}) + \ldots$$

that can be computed term by term.

[48]
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We define $\phi(x) = z$ as

$$
\phi(x) = \arg\min_z \pi_{oo}(x, \psi(z))
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$$\phi(x) = \arg\min_z \pi_{oo}(x, \psi(z))$$

so $\phi(x)$ satisfies

$$0 = \frac{\partial \pi_{oo}}{\partial \bar{x}}(x, \psi(\phi(x))) \frac{\partial \psi}{\partial z}(\phi(x))$$
From the choice of $\psi$,

$$\frac{\partial \psi}{\partial z}(\phi(x)) = \begin{bmatrix} I \\ 0 \end{bmatrix}$$
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so $\phi_i(x) = z_i$ satisfies for $1 \leq i \leq k$

$$\phi_i(x) = x_i + \frac{1}{\tau_i} \left( \frac{\partial \pi^3}{\partial \bar{x}_i}(x, (\phi(x), 0)) + \frac{\partial \pi^4}{\partial \bar{x}_i}(x, (\phi(x), 0)) + \ldots \right)$$
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This can be solved term by term via repeated substitution.

$$\phi_i^{(1)}(x) = x_i$$

$$\phi_i^{(2)}(x) = x_i + \frac{1}{\tau_i} \frac{\partial \pi_0^3}{\partial x_i}(x, (\phi^{(1)}(x), 0))$$

$$\vdots$$
Nonlinear Error Estimate

**Full Order Model** $x \in \mathbb{R}^n$

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{align*}
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**Reduced Order Model** \( z \in \mathbb{R}^k \)

\[
\begin{align*}
\dot{z} &= a(z, u) = \frac{\partial \phi}{\partial x}(\psi(z)) f(\psi(z), u) \\
y &= c(z) = h(\psi(z))
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\]

What is the error between their input-output maps?

What is the error between their Hankel maps?

What is the error between their Hankel maps restricted to optimal inputs?
Nonlinear Error Estimate

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Nonlinear Error Estimate

Full Order Optimal Feedback

\[ u = \kappa(x) = \left( \frac{\partial \pi_c}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) \right)' \]

\[ = G'x + \left( x' \frac{\partial f^{[2]}}{\partial u}(x, G'x) \right)' + \ldots \]

This can be solved term by term via repeated substitution.
Nonlinear Error Estimate

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Combined closed loop system

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
F + GG' & 0 \\
BG' & A
\end{bmatrix} \begin{bmatrix}
x \\
z
\end{bmatrix} + \ldots
\]
Nonlinear Error Estimate

Full Order Optimal Feedback

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F + GG' & 0 \\
BG' & A
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\begin{bmatrix}
x \\
z
\end{bmatrix} + \ldots
\]

\( F + GG' \) is antistable.

\( A \) is stable.
So there is an unstable manifold $z = \theta(x)$ satisfying
\[ a(\theta(x), \kappa(x)) = \frac{\partial\theta}{\partial x}(x)f(x, \kappa(x)) \]
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$$a(\theta(x), \kappa(x)) = \frac{\partial \theta}{\partial x}(x)f(x, \kappa(x))$$

This PDE can be solved term by term.
Nonlinear Error Estimate

Define the cross-observability function

\[ \rho(x^0, z^0) = \frac{1}{2} \int_0^\infty |y(t) - w(t)|^2 dt \]

where

\[
\begin{align*}
\dot{x} &= f(x, 0) \\
y &= h(x) \\
x(0) &= x^0 \\
\dot{z} &= a(z, 0) \\
w &= c(z) \\
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\end{align*}
\]

Then \( \rho \) satisfies the Lyapunov PDE

\[ 0 = \frac{\partial \rho}{\partial (x, z)}(x, z) \left[ \begin{array}{c} f(x, 0) \\ a(z, 0) \end{array} \right] + \frac{1}{2} |h(x) - c(z)|^2 \]

which can be solved term by term.
Nonlinear Error Estimate

Let \( u_x(-\infty : 0) \) be the optimal control that excites the full order system to \( x(0) = x \). Then the nonlinear Hankel maps satisfy

\[
|\mathcal{H}_n(u_x(-\infty : 0)) - \mathcal{H}_k(u_x(-\infty : 0))|^2 \leq \rho(x, \theta(x))
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\]

A good estimate of the error between the nonlinear Hankel maps is

\[
\sup \left( \frac{\rho(x, \theta(x))}{\pi_c(x)} \right)^{1/2}
\]
Nonlinear Error Estimate

As with input normal form we can make a change of coordinates so that

\[ \pi_c(x) = \frac{1}{2} |x|^2 + O(x)^{d+2} \]

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\( \epsilon_i^{[0:d-1]}(x_i) \) are the error polynomials of degrees 0 through \( d - 1 \).
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\[ \epsilon_i^{[0:d-1]}(x_i) \] are the error polynomials of degrees 0 through \( d - 1 \).

The error polynomials are unique for \( d \leq 6 \).
Nonlinear Example

Three linked rods connected by planar rotary joints with springs and dampening hanging from the ceiling. The input is a torque applied to the top joint and the output is the horizontal displacement of the bottom. Each rod is uniform of length $l = 2$, mass $\mu = 1$, with spring constant $c = 3$, dampening constant $b = 0.5$ and gravity constant $g = 0.5$. 
We approximated the nonlinear system by its Taylor series through terms of degree 5.
Nonlinear Example

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The Taylor series of controllability and observability functions $\pi_c(x), \pi_o(x)$ were computed through terms of degree 6.
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The system was brought into input normal form of degree 5 by a changes of state coordinates of degrees 1 through 5.
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The Hankel singular values of the linear part of the system are

\[
\begin{bmatrix}
15.3437 & 14.9678 & 0.3102 & 0.2470 & 0.0156 & 0.0091
\end{bmatrix}
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\]

Apparently only two dimensions are linearly significant.
Nonlinear Example

Here are the squared singular value polynomials.

\[
\begin{align*}
\tau_{1}^{[0:4]}(x_1) &= 235.4298 - 3.4163x_1^2 - 0.3104x_1^4 \\
\tau_{2}^{[0:4]}(x_2) &= 224.0356 - 3.2750x_2^2 - 0.2941x_2^4 \\
\tau_{3}^{[0:4]}(x_3) &= 000.0962 + 0.0014x_3^2 - 0.0001x_3^4 \\
\tau_{4}^{[0:4]}(x_4) &= 000.0610 + 0.0006x_4^2 + 0.0000x_4^4 \\
\tau_{5}^{[0:4]}(x_5) &= 000.0002 + 0.0000x_5^2 + 0.0000x_5^4 \\
\tau_{6}^{[0:4]}(x_6) &= 000.0001 + 0.0000x_6^2 + 0.0000x_6^4
\end{align*}
\]

Apparently only two dimensions are nonlinearly significant.
Nonlinear Example

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\tau_1^{[0:4]}(x_1) &= 235.4298 - 3.4163x_1^2 - 0.3104x_1^4 \\
\tau_2^{[0:4]}(x_2) &= 224.0356 - 3.2750x_2^2 - 0.2941x_2^4 \\
\tau_3^{[0:4]}(x_3) &= 000.0962 + 0.0014x_3^2 - 0.0001x_3^4 \\
\tau_4^{[0:4]}(x_4) &= 000.0610 + 0.0006x_4^2 + 0.0000x_4^4 \\
\tau_5^{[0:4]}(x_5) &= 000.0002 + 0.0000x_5^2 + 0.0000x_5^4 \\
\tau_6^{[0:4]}(x_6) &= 000.0001 + 0.0000x_6^2 + 0.0000x_6^4
\end{align*}
\]

Apparently only two dimensions are nonlinearly significant.
Nonlinear Example

Semilog plot of the squared singular value polynomials $\tau_i^{[0:4]}$

Notice the difference in scale and how flat they are.
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By way of comparison, the square of the third Hankel singular value is

\[
0.0962
\]

so this estimate is tight.
Nonlinear Example

Here are the error polynomials $\varepsilon_i^{[0:2]}$
Nonlinear Example

Here are outputs of the Hankel maps of the full and reduced systems excited by an optimal control $u_x(-\infty : 0)$ for random $x$. 

![Graph showing nonlinear output in blue and reduced nonlinear output in green]
Nonlinear Example

Here are the responses of the full nonlinear model (blue), the reduced nonlinear model (green) and the linear part of the full model (red) to a sinusoidal input.
Conclusion

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- We believe that these bounds are typical of the errors for inputs that are used in practice.
- The methodology extends directly to other methods of balancing such as $LQG$, $H_\infty$. 
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