Nonlinear Analysis of Phase Synchronization Systems: Phase-locked Loop and Costas Loop

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Content

- History of phase-locked loop (PLL)
- Methods for analysis and design of phase-locked loop
  - Simulation of phase-locked loop
  - Computation of self-excited attractors and hidden attractors (hidden periodic oscillations and hidden chaotic attractors)
  - Hidden attractors in applied models
  - Linear analysis
  - Nonlinear analysis of phase-locked loop
- Nonlinear analysis and simulation of classical PLL
  - Equivalence of PLL models in the time (signal) and phase-frequency spaces
  - Computation of multiplier phase detector characteristics
  - Averaging methods and Dynamical model of PLL
  - PLL simulation
- Nonlinear analysis and simulation of Costas loop
- Nonlinear analysis and simulation of classical DPLL
Phase-locked loops (PLL): history

- **Radio and TV**
  - de Bellescize H., La réception synchrone, L’Onde Électrique, 11, 1932

- **Computer architectures (frequency multiplication)**
  - Ian Young, PLL in a microprocessor i486DX2-50 (1992)
  - (in Turbo regime stable operation was not guaranteed)

- **Theory and Technology**
PLL in Computer architectures

Clock Skew Elimination: multiprocessor systems

Frequencies synthesis: multicore processors, motherboards
PLL operation: generation of electrical signal (voltage), phase of which is automatically tuned to phase of input reference signal (after transient process the signal, controlling the frequency of tunable osc., is constant)

Design: Signals class (sinusoidal, impulse...), PLL type (PLL, ADPLL, DPLL...)

Analysis: Choose PLL parameters (VCO, PD, Filter etc.) to achieve stable operation for the desired range of frequencies and transient time

Analysis methods: simulation, linear analysis, nonlinear analysis of mathematical models in signal space and phase-frequency space
D. Abramovitch, ACC-2008 plenary lecture:

*Full simulation of PLL in signal/time space is very difficult since one has to observe simultaneously very fast time scale of input signals and slow time scale of signal’s phase.*

How to construct model of signal’s phases?

**Simulation in phase space:** Could stable operation be guaranteed for all possible inputs, internal blocks states by simulation?

N. Gubar’ (1961), hidden oscillation in PLL

\[
\dot{\eta} = \alpha \eta - (1 - a \alpha)(\sin(\sigma) - \gamma), \\
\dot{\sigma} = \eta - a(\sin(\sigma) - \gamma)
\]

Global stability in simulation, but only bounded region of stability in reality
PLL simulation: attractors computation

**self-excited attractor localization:** *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it.

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \varepsilon(1-x^2)y \\
\end{align*}
\]

Van der Pol

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\sigma(x - y) \\
\dot{z} &= -bz + xy \\
\end{align*}
\]

Lorenz

\[
\begin{align*}
x &= -\sigma(x - y) \\
y &= rx - y - xz \\
z &= -bz + xy \\
\end{align*}
\]

hidden attractor: *if basin of attraction does not intersect with a small neighborhood of equilibria* [Leonov, Kuznetsov, Vagaitsev, *Phys. Lett. A*, 2011]

✓ *standard computational procedure* does not work: all equilibria are stable or not in the basin of attraction

✓ integration with random initial data does not work: basin of attraction is small, system’s dimension is large

How to choose initial data in the attraction domain?
PLL analysis & design: analytical linear methods

\[
\sin(\theta_1(t)) - \frac{1}{2}(\sin(\theta_1(t)-\theta_2(t)) + \sin(\theta_1(t)+\theta_2(t)))
\]

\[
\cos(\theta_2(t))
\]

\[
\text{PD} \quad \text{VCO} \quad g(t) \quad \text{Filter} \quad \text{F}(s)
\]

**Linearization:** while this is useful for studying loops that are near lock, it does not help for analyzing the loop when misphasing is large.

\[\otimes - \text{PD} : \sin(\theta_1) \cos(\theta_2) = 0.5 \sin(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2) \approx 0.5(\theta_1 + \theta_2)\]

\[\text{VCO} : \quad \dot{\theta}_2(t) = \omega_{\text{free}} + Lg(t)\]

**Linearization justification problems:**
Harmonic linearization (DFM) and Aizerman & Kalman problems,
Time-varying linearization & Perron effects of Lyapunov exponent sign inversions


PLL analysis & design: analytical nonlinear methods

PLL correct operation depends on the fact that it is nonlinear
PLL includes nonlinear blocks – phase-detector & VCO, which translate the consideration from signal response to phase response and back again

Stability analysis and design of the loops tends to be done by a combination of linear analysis, rule of thumb, and simulation. The experts in PLLs tend to be electrical engineers with hardware design backgrounds

To fill this gap it is necessary to develop and apply Nonlinear analysis and design of PLL

\[
\sin(\theta^1(t)) \quad 0.5(\sin(\theta^1(t) - \theta^2(t)) + \sin(\theta^1(t) + \theta^1(t))) \quad \cos(\theta(t))
\]

\[
\theta^1 \quad + \quad 0.5 \quad 0.5 \theta = (\theta^1 - \theta^2) \quad \theta^2 \quad VCO 
\]

\[
g(t) \quad \text{Filter} \quad g(t)
\]

\[
\theta^1 \quad g \quad \text{VCO} \quad g \quad F(s)
\]
Classical PLL: models in time & phase-freq. domains

1) Control signals in signals and phase-frequency spaces are equal.
2) Qualitative behaviors of signals and phase-freq. models are the same.

\[ \dot{\theta}^j \geq \omega_{\text{min}}, \quad |\dot{\theta}^1 - \dot{\theta}^2| \leq \Delta \omega, \quad |\dot{\theta}^j(t) - \dot{\theta}^j(\tau)| \leq \Delta \Omega, \quad |\tau - t| \leq \delta, \quad \forall \tau, t \in [0, T] \quad (*) \]

Waveforms:

\[ f^p(\theta) = \sum_{i=1}^{\infty} \left( a^p_i \sin(i\theta) + b^p_i \cos(i\theta) \right), \quad p = 1, 2 \]

\[ b^p_i = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \sin(ix) \, dx, \quad a^p_i = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \cos(ix) \, dx, \]

**Thm.** If (*)

\[ \varphi(\theta) = \frac{1}{2} \sum_{l=1}^{\infty} \left( (a^1_l a^2_l + b^1_l b^2_l) \cos(l\theta) + (a^1_l b^2_l - b^1_l a^2_l) \sin(l\theta) \right) \]

then

\[ |G(t) - g(t)| = O(\delta), \quad \forall t \in [0, T] \]

PD characteristics examples

\[ f^{1,2}(\theta) \]

\[ \varphi(\theta) \]

\[ f'(\theta(t)) \]

\[ f''(\theta(t)) \]

\[ \dot{x} = Ax + b \]

\[ g = c* x \]

\[ \dot{\theta}(t) = \omega^2_{\text{free}} + Lg(t) \]

\[ \varphi - \text{PD characteristic} \]

\[ f^{1,2} - \text{waveforms} \]
PD characteristics examples

\[ f^{1,2}(\theta) \]

\[ \varphi(\theta) \]

- \( f^{1,2}(\theta) \) — waveforms

- \( \varphi — \) PD characteristic

\[ \dot{\theta}(t) = \omega_{\text{free}} + Lg(t) \]

\[ \dot{x} = Ax + b f^2(\theta(t)) f'(\theta(t)) \]

\[ g = c^*x \]

\[ f^2(\theta(t)) \]

\[ VCO \]

\[ \varphi(\theta^2 - \theta') \]

\[ \dot{x} = Ax + b \varphi(\theta^2 - \theta') \]

\[ G = c^*x \]

\[ \theta^2(t) \]

\[ VCO \]
Differential equations of PLL

- master oscillator phase: $\dot{\theta}_1(t) = \omega_1$
- phase & freq. difference: $\theta_d(t) = \theta_1(t) - \theta_2(t), \omega_2 - \omega_1 = \Delta\omega$

Nonautonomous equations

$$\begin{align*}
\dot{x} &= Ax + b f_1(\omega_1 t) f_2(\theta_d + \omega_1 t), \\
\dot{\theta}_d &= (\omega_2 - \omega_1) + Lc^* x, \\
x(0) &= x_0, \quad \theta_d(0) = \theta_0
\end{align*}$$

Autonomous equations

$$\begin{align*}
\dot{x} &= Ax + b \varphi(\theta_d), \\
\dot{\theta}_d &= \Delta\omega + Lc^* x, \\
x(0) &= x_0, \quad \theta_d(0) = \theta_0
\end{align*}$$

?) Qualitative behaviors of signals and phase-freq. models are the same.
 Autonomous differential equations of PLL

[?] Qualitative behaviors of signals and phase-freq. models are the same.

\[ \dot{z} = Az + bf^1(\omega^1 t) f^2(\eta_d + \omega^1 t), \quad \Leftrightarrow \quad \dot{x} = Ax + b\varphi(\theta_d) \]

\[ \dot{\eta}_d = \omega_d + Lc^*z \]

\[ \dot{\theta}_d = \omega_d + Lc^*x \]

If \( \omega^1(t) \equiv \omega^1 \) then averaging method [Bogolubov, Krylov] allows one to justify passing to autonomous diff. equations of PLL.

\[ \tau = \omega^1 t, \quad \omega^1 \frac{du}{d\tau} = Au + bf^1(\tau) f^2(\eta_d + \tau), \]

\[ u(\tau) = z \left( \frac{\tau}{\omega^1} \right) = z(t), \quad \omega^1 \frac{d\eta_d}{d\tau} = \omega_d + Lc^*u. \]

Aver. method requires to prove

\[ \frac{1}{T} \int_0^T f^1(\tau) f^2(\eta_d + \tau) - \varphi(\eta_d) dt \Rightarrow 0 \quad as \quad T \to \infty \]

(it is satisfied for phase detector characteristic \( \varphi \) computed above).

Solutions \((z(t), \eta_d(t)) \& (x(t), \theta_d(t))\) (with the same initial data)) of nonautonomous and autonomous models are close to each other on a certain time interval as \( \omega^1 \to \infty \).
Nonlinear analysis of PLL

Continuous
\[ \dot{x} = Ax + b\varphi(\theta_d), \]
\[ \dot{\theta}_d = \Delta \omega + Lc^* x, \]
\[ x(0) = x_0, \quad \theta_d(0) = \theta_0 \]

Discrete
\[ x(t + 1) = Ax(t) + b\varphi(\theta_d(t)), \]
\[ \theta_d(t + 1) = \theta_d(t) + \Delta \omega + Lc^* x(t), \]
\[ x(0) = x_0, \quad \theta_d(0) = \theta_0 \]

Here it is possible to apply various well developed methods of mathematical theory of phase synchronization

- V. Yakubovich, G. Leonov, A. Gelig, Stability of Systems with Discontinuous Nonlinearities, (Singapore: World Scientific), 2004
PLL simulation: Matlab Simulink

0.2 seconds

REF frequency

Integrator

Filter

PD

Out 1

VCO

20 seconds

REF frequency

Integrator

Interpreted MATLAB Fcn

Multiplication

Filter

VCO

Kuznetsov, Leonov et al., Patent 2449463, 2011
Kuznetsov, Leonov et al., Patent 112555, 2011
Costas loop: digital signal demodulation


- signal demodulation in digital communication

\[ m(t) = \pm 1 \text{ — data,} \]
\[ m(t) \sin(2\omega t) \text{ and } m(t) \cos(2\omega t) \]
- can be filtered out,
\[ m(t) \cos(0) = m(t), \]
\[ m(t) \sin(0) = 0 \]

- wireless receivers
- Global Positioning System (GPS)
Costas Loop: PD characteristic computation

Analysis of Costas Loop can be reduced to the analysis of PLL:

\[ u^1(\theta^1(t)) = f^1(\theta^1(t)) f^1(\theta^1(t)), \]
\[ u^2(\theta^2(t)) = f^2(\theta^2(t)) f^2(\theta^2(t) - \frac{\pi}{2}), \]

Costas Loop PD characteristic: \( A^k_l, B^k_l \) – Fourier coefficients of \( u_k(\theta) \)

\[ u(\theta(t)) = f^i(\theta^i(t)) f^i(\theta^i(t)) \]
\[ u(\theta(t)) = f^i(\theta^i(t)) f^i(\theta^i(t) - 90^\circ) \]

\[ \varphi(\theta) = \frac{A^0_1 A^0_2}{4} + \frac{1}{2} \sum_{l=1}^{\infty} \left( (A^l_1 A^l_2 + B^l_1 B^l_2) \cos(l\theta) + (A^l_1 B^l_2 - B^l_1 A^l_2) \sin(l\theta) \right) \]

Then \( |G(t) - g(t)| \leq C\delta, \forall t \in [0, T] \)

Theorem. If (1)–(2) and

\[ f^{1,2}(\theta) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\theta) \]

\[ \varphi(\theta) = -\frac{1}{72} + \frac{1}{2} \sum_{l=1}^{\infty} \begin{cases} 
\frac{16}{\pi^4 l^4} \cos(l\theta), & l = 4p, \\
-\frac{8}{\pi^3 l^3} \sin(l\theta), & l = 4p + 2, \\
-\frac{4(\pi l-2)}{\pi^4 l^4} \cos(l\theta) - \frac{4(\pi l-2)}{\pi^4 l^4} \sin(l\theta), & l = 4p + 1, \\
\frac{4(\pi l+2)}{\pi^4 l^4} \cos(l\theta) - \frac{4(\pi l+2)}{\pi^4 l^4} \sin(l\theta), & l = 4p + 3
\end{cases} \]
Costas loop simulation: Matlab Simulink

Kuznetsov N.V. Nonlinear analysis of Phase-locked loop and Costas Loop, CityU HK (2012) 20 / 29
Squaring loop

PLL-based carrier recovery circuit
Golomb et al., 1963; Stiffler, 1964; Lindsey, 1966

- wireless receivers
- demodulation

\[ m(t) = \pm 1 \quad \text{data}, \]
\[ (m(t)\sin(\omega t))^2 = \frac{1 - \cos(2\omega t)}{2} \quad \text{— squaring allows to remove data} \]
\[ 0.5 \cos(2\omega t) \quad \text{— filter removes the constant} \]
**Classical DPLL: nonlinear analysis**

\[ \phi_{k+1} - (\phi_k + 2\pi) = 2\pi \left( \frac{\omega_{in}}{\omega} - 1 \right) - \frac{\omega_{in}}{\omega} \omega G \sin(\phi_k) \]

\[ \omega_{in} = \omega, \phi_k = \omega t_k + \theta_0, \quad \phi_k + 2\pi \rightarrow \sigma_k \in [\pi, \pi] \]

\[ \sigma_{t+1} = \sigma_t - r \sin \sigma_t, \quad r = \omega G \]

- **\( r_1 = 2 \):** Global asymptotic stability disappears and appears unique and globally stable on \((-\pi, 0)\cup(0, -\pi)\) cycle with period 2
- **\( r_2 = \pi \):** Bifurcation of splitting: unique cycle loses stability and two locally stable asymmetrical cycles with period 2 appear
- **\( r_3 = \sqrt{\pi^2 + 2} \):** Two locally stable asymmetrical cycles with period 2 lose stability and two locally stable asymmetrical cycles with period 4 appear

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Classical DPLL: simulation of transition to chaos

<table>
<thead>
<tr>
<th>№</th>
<th>Bifurcation value of parameter, ( r_j )</th>
<th>Feigenbaum const ( \delta_j = \frac{r_j - r_j - 1}{r_j + 1 - r_j} )</th>
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<tr>
<td>11</td>
<td>3.531521154835959</td>
<td></td>
</tr>
</tbody>
</table>

Feigenbaum const for unimodal maps:
\[ \lim_{n \to +\infty} \delta_n = 4.669 \ldots \]


\[ f(x) = x - 2.2 \sin(x) \]

nonunimodal map
Publications: PLL, 2012-2011

Publications: Hidden attractors


Lyapunov exponent: chaos, stability, Perron effects, linearization

Time-varying linearization without justification may lead to wrong results

\[
\begin{cases}
\dot{x} = F(x), \ x \in \mathbb{R}^n, \ F(x_0) = 0 \\
x(t) \equiv x_0, \ A = \frac{dF(x)}{dx} \Big|_{x=x_0}
\end{cases}
\begin{cases}
\dot{y} = Ay + o(y) \\
y(t) \equiv 0, \ (y = x - x_0)
\end{cases}
\begin{cases}
\dot{z} = Az \\
z(t) \equiv 0
\end{cases}
\]

✓ stationary: \( z(t) = 0 \) is exp. stable \( \Rightarrow \) \( y(t) = 0 \) is asympt. stable

? nonstationary: \( z(t) = 0 \) is exp. stable \( \Rightarrow ? \) \( y(t) = 0 \) is asympt. stable

! Perron effect: \( z(t) = 0 \) is exp. stable (unst), \( y(t) = 0 \) is exp. unstable (st)

Positive largest Lyapunov exponent doesn’t, in general, indicate chaos

Hidden oscillations in control: Aizerman & Kalman problem:  
Harmonic linearization without justification may lead to wrong results

Harmonic balance & Describing function method in Absolute Stability Theory

\[ \dot{x} = P x + q \psi(r^* x), \quad \psi(0) = 0 \quad (1) \]

\[ W(p) = r^* (P - pI)^{-1} q \]

\[ \text{Im} W(i\omega_0) = 0, \quad k = - (\text{Re} W(i\omega_0))^{-1} \]

DFM: exists periodic solution \( \sigma(t) = r^* x(t) \approx a \cos \omega_0 t \)

\[ a : \quad \int_0^{2\pi/\omega_0} \psi(a \cos \omega_0 t) \cos \omega_0 t \, dt = ka \int_0^{2\pi/\omega_0} (\cos \omega_0 t)^2 \, dt \]

Aizerman problem: If (1) is stable for any linear \( \psi(\sigma) = \mu \sigma, \quad \mu \in (\mu_1, \mu_2) \)
then (1) is stable for any nonlinear \( \psi(\sigma) : \mu_1 \sigma < \psi(\sigma) < \mu_2 \sigma, \quad \forall \sigma \neq 0 \)

DFM: (1) is stable \( \Rightarrow k : \quad k < \mu_1, \mu_2 < k \Rightarrow k \sigma^2 < \psi(\sigma) \sigma, \quad \psi(\sigma) \sigma < k \sigma^2 \)
\( \Rightarrow \forall a \neq 0 : \quad \int_0^{2\pi/\omega_0} (\psi(a \cos \omega_0 t) a \cos \omega_0 t - k(a \cos \omega_0 t)^2) \, dt \neq 0 \)
\( \Rightarrow \) no periodic solutions by harmonic linearization and DFM, but

Hidden attractor in classical Chua’s system

In 2010 the notion of *hidden attractor* was introduced and hidden chaotic attractor was found for the first time by the authors [Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Physics Letters A, 375(23), 2011]

\[
\begin{align*}
\dot{x} &= \alpha(y - x - m_1 x - \psi(x)) \\
\dot{y} &= x - y + z, \quad \dot{z} = -(\beta y + \gamma z) \\
\psi(x) &= (m_0 - m_1) \text{sat}(x)
\end{align*}
\]

\[\alpha = 8.4562, \quad \beta = 12.0732\]
\[\gamma = 0.0052\]
\[m_0 = -0.1768, \quad m_1 = -1.1468\]

equilibria: stable zero \(F_0\) & 2 saddles \(S_{1,2}\)
trajectories: 'from' \(S_{1,2}\) tend (black) to zero \(F_0\) or tend (red) to infinity;
Hidden chaotic attractor (in green) with positive Lyapunov exponent